

Delocalization of a $(1 + 1)$ -dimensional stochastic wave equation*

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Version: October 24, 2016

Abstract

A noteworthy property of many parabolic stochastic PDEs is that they locally linearize [3, 4, 6, 7]. We prove that, by contrast, a large family of stochastic wave equations in dimension one do not possess this important property.

Keywords. Stochastic wave equation, quadratic variation, localization, central limit theorem, the law of iterated logarithm.

AMS 2010 subject classification. Primary 60H15; Secondary 35R60, 60G60.

1 Introduction

Consider the following stochastic partial differential equation (SPDE, for short), indexed by space-time, $(0, \infty) \times \mathbb{R}$:

$$\partial_t v = \partial_{xx}^2 v + \sigma(v)\xi \quad \text{subject to} \quad v(0) \equiv 1; \quad (1)$$

where $v = v(t, x)$ for all space-time points $(t, x) \in (0, \infty) \times \mathbb{R}$; the forcing term ξ denotes space-time white noise; and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a nonrandom, Lipschitz continuous function. It is well known in this, and related, contexts, that the solution to (1) locally linearizes. Indeed, let $Z = Z(t, x)$ denote the linearization of v ; that is, Z solves the SPDE (1) with $\sigma \equiv 1$. Then,

$$v(t, x + \varepsilon) - v(t, x) = \sigma(v(t, x))(Z(t, x + \varepsilon) - Z(t, x)) + \mathcal{R}_{t,x}(\varepsilon),$$

where, as $\varepsilon \downarrow 0$, the remainder term $\mathcal{R}_{t,x}(\varepsilon)$ tends to zero much faster than $Z(t, x + \varepsilon) - Z(t, x)$ does. This fact appears explicitly—in different senses and contexts—in [3, 6, 7], and in a different setting earlier in the fundamental works of Hairer [4, 5] respectively on the KPZ equation and on Hairer’s theory of regularity structures.

*Research supported in part by the NSF grants DMS-1307470.

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The purpose of this short note is to point out that, in sharp contrast with the parabolic setting, typical hyperbolic SPDEs do not locally linearize. To be concrete, let us consider a hyperbolic SPDE of the type,

$$\partial_{tt}^2 u = \partial_{xx}^2 u + \sigma(u)\xi \quad \text{on } (0, \infty) \times \mathbb{R}, \quad (2)$$

subject to the initial conditions $u(0) = 0$ and $(\partial_t u)(0) = 1$, to be concrete. Let Y denote the linearization of u ; that is, let Y denote the solution to (2) with $\sigma \equiv 1$.

Let us suppose, to the contrary, that u locally linearizes; that is, let us posit that

$$u(t, x+h) - u(t, x) = \sigma(u(t, x)) (Y(t, x+h) - Y(t, x)) + \tilde{\mathcal{R}}_{t,x}(h),$$

where the remainder term $\tilde{\mathcal{R}}_{t,x}(h)$ is significantly smaller than $Y(t, x+h) - Y(t, x)$ when $h \approx 0$. Then, a simple heuristic argument would suggest that, for every fixed $t > 0$, the quadratic variation of $[x_1, x_2] \ni x \mapsto u(t, x)$ would have to be a function of $u(t)$ alone; in fact, a more careful heuristic analysis of the random field Y might suggest that the quadratic variation of $u(t)$ is likely to be equal to $2t \int_{x_1}^{x_2} [\sigma(u(t, x))]^2 dx$ at every fixed time point $t > 0$. Theorem 1 refutes this assertion, and hence rules out the possibility of the existence of good local linearizations to u .

Theorem 1. *Choose and fix four real numbers $t > 0$ and $x, X_1, X_2 \in \mathbb{R}$ such that $X_1 < x < X_2$. For all integers $N \geq 1$ and $1 \leq i \leq N$, define $t_i = t_{i,N} := (i-1)t/N$ and $x_i = x_{i,N} := X_1 + (i-1)(X_2 - X_1)/N$ for integers $1 \leq i \leq N$. Then, the following is valid for every real number $p \geq 2$. As $N \rightarrow \infty$:*

$$\sum_{i=1}^N [u(t_{i+1}, x) - u(t_i, x)]^2 \xrightarrow{\text{in } L^p(\Omega)} \int_{Q(x,t)} [\sigma(u(s - |x - y|, y))]^2 ds dy; \quad \text{and} \quad (3)$$

$$\sum_{i=1}^N [u(t, x_{i+1}) - u(t, x_i)]^2 \xrightarrow{\text{in } L^p(\Omega)} \int_{(-t,t) \times (X_1, X_2)} [\sigma(u(s, y - |t - s|))]^2 ds dy; \quad (4)$$

where $Q(x, t) := \{(s, y) : |y - x| \leq |t - s|\}$.

The proof of (3) hinges on the observation that $u(r, y) \approx u(t_i - |x - y|, y)$ uniformly for all space-time points

$$(r, y) \in Q(x, t_{i+1}, t_i) := Q(x, t_{i+1}) \setminus Q(x, t_i); \quad (5)$$

see Section 2. This kind of approximation procedure also yields the following limit theorem as a by product. In order to state the next result, first let $\mathcal{G}_{t,x}^0$ denote the P-completion of the σ -algebra generated by all Wiener integrals of the form $\int \phi d\xi$, where ϕ is smooth and support in $Q(x, t)$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then, define

$$\mathcal{G}_{t,x} := \bigcap_{s>t} \mathcal{G}_{s,x}^0 \quad \text{for every } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Theorem 2. Choose and fix $t > 0$ and $x \in \mathbb{R}$. Then,

$$\frac{u(t + \varepsilon, x) - u(t, x)}{\sqrt{\varepsilon}} \xrightarrow{d} \int_{x-t}^{x+t} \sigma(u(t - |x - y|, y)) W(dy) \quad \text{as } \varepsilon \downarrow 0, \quad (6)$$

where W is a standard two-sided Brownian motion that is independent of $\mathcal{G}_{t,x}$. Moreover,

$$\limsup_{\varepsilon \downarrow 0} \frac{u(t + \varepsilon, x) - u(t, x)}{\sqrt{2\varepsilon \log \log(1/\varepsilon)}} = \sqrt{\int_{x-t}^{x+t} [\sigma(u(t - |x - y|, y))]^2 dy} \quad \text{a.s.} \quad (7)$$

Theorems 1 and 2 are proved respectively in Sections 2 and 3.

2 Proof of Theorem 1

First let us recall that the random field $u = \{u(t, x)\}_{t \geq 0, x \in \mathbb{R}}$ is a mild solution to (2); that is:

- $\{u(t)\}_{t \geq 0}$ is predictable with respect to the Brownian filtration generated by all Wiener integrals of the form $\int_{[0,t] \times \mathbb{R}} \phi d\xi$ for $\phi \in L^2(\mathbb{R}_+ \times \mathbb{R})$; and
- For each $t \geq 0$ and $x \in \mathbb{R}$,

$$u(t, x) = 1 + \int_{Q(x,t)} \sigma(u(s, y)) \xi(ds dy) \quad \text{a.s.}, \quad (8)$$

where $Q(x, t)$ is defined in Theorem 1, and the stochastic integral is understood in the sense of Walsh [9].

It is well known that the methods of Walsh's theory imply that (2) is well posed in the sense that there exists a unique continuous solution to the integral equation (2). It is also well known that for every $k \geq 2$,

$$\mathbb{E} (|u(t_1, x_1) - u(t_2, x_2)|^k) \lesssim |t_1 - t_2|^{k/2} + |x_1 - x_2|^{k/2}, \quad (9)$$

uniformly for all $t_1, t_2 \in [0, t]$ and $x_1, x_2 \in \mathbb{R}$; consult Peszat and Zabczyk [8]. This result appears, implicitly, earlier in Dalang and Frangos [2]. Moreover, nearly all of the preceding, and much more, is included in the general theories of Dalang [1] and Peszat and Zabczyk [8].

We will prove (3) and (4) in successive steps, and in this order.

Step 1. Recall the sets $Q(x, t_{i+1}, t_i)$, defined earlier in (5), and set

$$\mathcal{A}_N(x) := \sum_{i=1}^N [u(t_{i+1}, x) - u(t_i, x)]^2 \quad \& \quad \mathcal{B}_N(x) := \sum_{i=1}^N \left(\int_{Q(x, t_{i+1}, t_i)} \sigma(u(r_i(y), y)) \xi(ds dy) \right)^2,$$

where $r_i(y) := \max(t_i - |x - y|, 0)$.

In this first step of the proof, we will prove that $\mathcal{A}_N(x) - \mathcal{B}_N(x) \rightarrow 0$ in $L^p(\Omega)$ as $N \rightarrow \infty$. To this end, let us first observe that, for all $1 \leq i \leq N$, the mild formulation of the solution

u yields the representation, $u(t_{i+1}, x) - u(t_i, x) = \int_{Q(x, t_{i+1}, t_i)} \sigma(u(s, y)) \xi(ds dy)$. Therefore, if we write $\|\cdot\|_p$ for the $L^p(\Omega)$ -norm, then $\|\mathcal{A}_N(x) - \mathcal{B}_N(x)\|_p$ is bounded above by

$$\sum_{i=1}^N \left\| \left[\int_{Q(x, t_{i+1}, t_i)} \sigma(u(s, y)) \xi(ds dy) \right]^2 - \left[\int_{Q(x, t_{i+1}, t_i)} \sigma(u(r_i(y), y)) \xi(ds dy) \right]^2 \right\|_p \leq \sum_{i=1}^N I_i J_i,$$

where

$$I_i := \left\| \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(s, y)) - \sigma(u(r_i(y), y))] \xi(ds dy) \right\|_{2p}, \text{ and}$$

$$J_i := \left\| \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(s, y)) + \sigma(u(r_i(y), y))] \xi(ds dy) \right\|_{2p}.$$

We estimate I_i first as follows: Because $|s - r_i(y)| \leq (t_{i+1} - t_i)$ uniformly for all $(s, y) \in Q(x, t_{i+1}, t_i)$, (9), the Burkholder–Davis–Gundy inequality, and Minkowski's inequality together yield

$$I_i^2 \lesssim \int_{Q(x, t_{i+1}, t_i)} \|\sigma(u(s, y)) - \sigma(u(r_i(y), y))\|_{2p}^2 ds dy \lesssim \int_{Q(x, t_{i+1}, t_i)} (t_{i+1} - t_i) ds dy,$$

uniformly for all $i = 1, \dots, N$. Thus, $I_i \lesssim t_{i+1} - t_i$ uniformly for all $i = 1, \dots, N$. Similarly,

$$J_i^2 \leq \int_{Q(x, t_{i+1}, t_i)} \|\sigma(u(s, y)) + \sigma(u(r_i(y), y))\|_{2p}^2 ds dy \lesssim t_{i+1} - t_i,$$

uniformly for all $i = 1, \dots, N$. These estimates of I_i and J_i can be combined to yield

$$\|\mathcal{A}_N(x) - \mathcal{B}_N(x)\|_p \lesssim \sum_{i=1}^N (t_{i+1} - t_i)^{3/2} \lesssim N^{-1/2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This concludes Step 1.

Step 2. Define

$$\mathcal{C}_N(x) := \sum_{i=1}^N \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(r_i(y), y))]^2 ds dy.$$

We plan to establish next that $\mathcal{B}_N(x) - \mathcal{C}_N(x) \rightarrow 0$ in $L^p(\Omega)$ as $N \rightarrow \infty$. According to the general theory of Dalang [1],

$$\sup_{s \in [0, t]} \sup_{x \in \mathbb{R}} \mathbb{E}(|u(s, x)|^q) < \infty \quad \text{for all } q \geq 2. \quad (10)$$

Therefore, the Burkholder–Davis–Gundy inequality ensures that the stochastic processes $\{\mathcal{B}_N(x)\}_{N \geq 1}$ and $\{\mathcal{C}_N(x)\}_{N \geq 1}$ are both bounded in $L^q(\Omega)$ for all $q \geq 2$. Consequently, it suffices to show that $\mathcal{B}_N(x) - \mathcal{C}_N(x) \rightarrow 0$ in $L^2(\Omega)$ as $N \rightarrow \infty$. With this aim in mind, write

$$\mathbb{E}(|\mathcal{B}_N(x) - \mathcal{C}_N(x)|^2) = B_1 + B_2 - 2B_3,$$

where

$$\begin{aligned} B_1 &:= \mathbb{E} \left[\left(\sum_{i=1}^N \left| \int_{Q(x, t_{i+1}, t_i)} \sigma(u(r_i(y), y)) \xi(ds dy) \right|^2 \right)^2 \right], \\ B_2 &:= \mathbb{E} \left[\left(\sum_{i=1}^N \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(r_i(y), y))]^2 ds dy \right)^2 \right], \quad \text{and} \\ B_3 &:= \mathbb{E} \left(\sum_{i=1}^N \left| \int_{Q(x, t_{i+1}, t_i)} \sigma(u(r_i(y), y)) \xi(ds dy) \right|^2 \times \sum_{i=1}^N \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(r_i(y), y))]^2 ds dy \right). \end{aligned}$$

In order to simplify the notation, define for all $i = 1, \dots, N$,

$$Q_i := \int_{Q(x, t_{i+1}, t_i)} \sigma(u(r_i(y), y)) \xi(ds dy), \quad \tilde{Q}_i := \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(r_i(y), y))]^2 ds dy.$$

Whenever $j < i$,

$$\mathbb{E} [Q_i^2 Q_j^2] = \mathbb{E} [\mathbb{E} (Q_i^2 Q_j^2 \mid \mathcal{G}_{t_i, x})] = \mathbb{E} [Q_j^2 \mathbb{E} (Q_i^2 \mid \mathcal{G}_{t_i, x})] = \mathbb{E} [\tilde{Q}_i Q_j^2].$$

Consequently,

$$B_1 = \mathbb{E} \sum_{1 \leq i, j \leq N} Q_i^2 Q_j^2 = \mathbb{E} \sum_{i=1}^N Q_i^4 + 2\mathbb{E} \sum_{1 \leq j < i \leq N} \tilde{Q}_i Q_j^2.$$

The same conditioning technique yields

$$B_3 = \mathbb{E} \sum_{1 \leq j < i \leq N} Q_j^2 \tilde{Q}_i + \mathbb{E} \sum_{1 \leq j < i \leq N} \tilde{Q}_j \tilde{Q}_i + \mathbb{E} \sum_{i=1}^N Q_i^2 \tilde{Q}_i.$$

As a result, it follows that

$$\mathbb{E} (|\mathcal{B}_N(x) - \mathcal{C}_N(x)|^2) = \mathbb{E} \sum_{i=1}^N Q_i^4 + \mathbb{E} \sum_{i=1}^N \tilde{Q}_i^2 - 2\mathbb{E} \sum_{i=1}^N Q_i^2 \tilde{Q}_i.$$

Thanks to the uniform boundedness of the moments, and the Burkholder–Davis–Gundy inequality,

$$\mathbb{E} (|\mathcal{B}_N(x) - \mathcal{C}_N(x)|^2) \lesssim \sum_{i=1}^N (t_{i+1} - t_i)^2 \lesssim N^{-1} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This concludes Step 2.

Step 3. We are ready to verify (3). Define

$$\mathcal{D}(x) := \int_{Q(x, t)} [\sigma(u(s - |x - y|, y))]^2 ds dy = \sum_{i=1}^N \int_{Q(x, t_{i+1}, t_i)} [\sigma(u(s - |x - y|, y))]^2 ds dy.$$

In light of Steps 1 and 2, it remains to prove that $\mathcal{C}_N(x) \rightarrow \mathcal{D}(x)$ in $L^p(\Omega)$ as $N \rightarrow \infty$. One can recycle the argument of Step 1 in order to the uniform-in- N bound,

$$\|\mathcal{C}_N(x) - \mathcal{D}(x)\|_p \lesssim \sum_{i=1}^N (t_{i+1} - t_i)^{3/2} \lesssim N^{-1/2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This completes Step 3, and hence the proof of (3).

Step 4. (Sketch) In this step we outline the proof of the remaining assertion (4) of Theorem 1. The details require considerably-more space, yet not many more ideas, than those in Steps 1–3.

For every $1 \leq i \leq N$ and for each x_i , let

$$L_i(t) := Q(x_i, t) \setminus Q(x_{i+1}, t) \quad \text{and} \quad R_i(t) := Q(x_{i+1}, t) \setminus Q(x_i, t).$$

By (8),

$$u(t, x_{i+1}) - u(t, x_i) = \int_{R_i(t)} \sigma(u(s, y)) \xi(ds dy) - \int_{L_i(t)} \sigma(u(s, y)) \xi(ds dy). \quad (11)$$

Define

$$\begin{aligned} A_N(t) &:= \sum_{i=1}^N [u(t, x_{i+1}) - u(t, x_i)]^2, \\ B_N(t) &:= \sum_{i=1}^N \left[\int_{R_i(t)} \sigma(u(v_i(y), y)) \xi(ds dy) - \int_{L_i(t)} \sigma(u(v_i(y), y)) \xi(ds dy) \right]^2, \\ C_N(t) &:= \sum_{i=1}^N \int_{L_i(t) \cup R_i(t)} [\sigma(u(v_i(y), y))]^2 ds dy, \text{ and} \\ D(t) &:= \int_{(0, t) \times (X_1, X_2)} ([\sigma(u(s, x - t + s))]^2 + [\sigma(u(s, x + t - s))]^2) ds dx, \end{aligned}$$

where $v_i(y) := \max(t + y - x_{i+1}, 0)$ if $(s, y) \in L_i(t)$, and $v_i(y) := \max(t - y + x_i, 0)$ if $(s, y) \in R_i(t)$. It is possible to adapt the arguments of Steps 1–3 in order to prove that $A_N(t) - B_N(t) \rightarrow 0$ in $L^p(\Omega)$ as $N \rightarrow \infty$. Next we argue that $B_N(t) - C_N(t) \rightarrow 0$ in $L^p(\Omega)$ as $N \rightarrow \infty$. It is easy to see that both $\{B_N(t)\}_{N \geq 1}$ and $\{C_N(t)\}_{N \geq 1}$ are uniformly bounded in $L^p(\Omega)$. Therefore, it suffices to prove the convergence of $B_N(t) - C_N(t)$ to zero in $L^2(\Omega)$.

In order to save on typography, define

$$\begin{aligned} R_i &:= \int_{R_i(t)} \sigma(u(v_i(y), y)) \xi(ds dy), \quad \hat{R}_i := \int_{R_i(t)} [\sigma(u(v_i(y), y))]^2 ds dy, \\ L_i &:= \int_{L_i(t)} \sigma(u(v_i(y), y)) \xi(ds dy), \quad \hat{L}_i := \int_{L_i(t)} [\sigma(u(v_i(y), y))]^2 ds dy. \end{aligned}$$

An expansion of the square yields

$$\mathbb{E}(|B_N(t) - C_N(t)|^2) = \mathbb{E} \left(\left| \sum_{i=1}^N (R_i - L_i)^2 - \sum_{i=1}^N (\hat{R}_i + \hat{L}_i) \right|^2 \right) := S_1 + S_2 - 2S_3,$$

where

$$S_1 := \mathbb{E} \left(\left| \sum_{i=1}^N (R_i - L_i)^2 \right|^2 \right) \quad \text{and} \quad S_2 := \mathbb{E} \left(\left| \sum_{i=1}^N (\hat{R}_i + \hat{L}_i) \right|^2 \right),$$

and S_3 is the remainder. We compute S_1 , S_2 , and S_3 in this order.

Let us introduce σ -algebras $\mathcal{F}_\pm(x_i)$ as follows: Let $\mathcal{F}_+(x_i, \varepsilon)$ denote the σ -algebra generated by $\int \phi d\xi$ for all smooth functions ϕ that are supported in $\cup_{x_1 \leq x \leq x_i + \varepsilon} Q(x, t)$. Similarly, let $\mathcal{F}_-(x_i, \varepsilon)$ denote the σ -algebra generated by $\int \phi d\xi$ for all smooth ϕ supported on $\cup_{x_i - \varepsilon \leq x \leq x_N} Q(x, t)$. Then, we define $\mathcal{F}_\pm(x_i) := \cap_{\varepsilon > 0} \mathcal{F}_\pm(x_i, \varepsilon)$,

If $1 \leq j < i \leq N$, then we may condition on $\mathcal{F}_+(x_i)$ and/or $\mathcal{F}_-(x_{j+1})$ in order to see that

$$\mathbb{E} [(R_i - L_i)^2 (R_j - L_j)^2] = \mathbb{E} [R_i^2 R_j^2 + L_i^2 R_j^2 + R_i^2 L_j^2 + L_i^2 L_j^2].$$

Similar considerations show that the above also holds when $1 \leq i < j \leq N$. Thus,

$$S_1 = \mathbb{E} \sum_{1 \leq i \neq j \leq N} (R_i^2 R_j^2 + L_i^2 R_j^2 + R_i^2 L_j^2 + L_i^2 L_j^2) + \mathbb{E} \sum_{i=1}^N (R_i - L_i)^4.$$

If $1 \leq j < i \leq N$, another conditioning argument yields

$$\mathbb{E} [R_i^2 R_j^2 + L_i^2 R_j^2 + R_i^2 L_j^2 + L_i^2 L_j^2] = \mathbb{E} [R_j^2 \hat{R}_i + L_i^2 R_j^2 + \hat{R}_i \hat{L}_j + L_i^2 \hat{L}_j].$$

By comparison, if $1 \leq i < j \leq N$,

$$\mathbb{E} [R_i^2 R_j^2 + L_i^2 R_j^2 + R_i^2 L_j^2 + L_i^2 L_j^2] = \mathbb{E} [R_i^2 \hat{R}_j + \hat{L}_i \hat{R}_j + R_i^2 L_j^2 + \hat{L}_i L_j^2].$$

Thus, we can rearrange the sum to see that

$$S_1 = 2\mathbb{E} \sum_{1 \leq j < i \leq N} (\hat{R}_i R_j^2 + \hat{L}_j \hat{R}_i + L_i^2 R_j^2 + L_i^2 \hat{L}_j) + \mathbb{E} \sum_{i=1}^N (R_i - L_i)^4.$$

For S_2 , there is no need for conditioning arguments, as a direct calculation yields

$$S_2 = 2\mathbb{E} \sum_{1 \leq j < i \leq N} (\hat{R}_i \hat{R}_j + \hat{L}_i \hat{R}_j + \hat{R}_i \hat{L}_j + \hat{L}_i \hat{L}_j) + \mathbb{E} \sum_{i=1}^N (\hat{R}_i + \hat{L}_i)^2.$$

Finally, another conditioning argument shows that

$$\begin{aligned} S_3 = \mathbb{E} \sum_{1 \leq j < i \leq N} & \left(\hat{R}_i \hat{R}_j + L_i^2 \hat{R}_j + \hat{R}_i \hat{L}_j + L_i^2 \hat{L}_j + R_j^2 \hat{R}_i + \hat{L}_j \hat{R}_i + R_j^2 \hat{L}_i + \hat{L}_i \hat{L}_j \right) \\ & + \mathbb{E} \sum_{i=1}^N (R_i - L_i)^2 (\hat{R}_i + \hat{L}_i). \end{aligned}$$

One can combine the preceding and compute to see, after a few lines, that

$$\begin{aligned} S_1 + S_2 - 2S_3 &= 2\mathbb{E} \sum_{1 \leq j < i \leq N} (L_i^2 - \hat{L}_i)(R_j^2 - \hat{R}_j) + \mathbb{E} \sum_{i=1}^N (R_i - L_i)^4 + \mathbb{E} \sum_{i=1}^N (\hat{R}_i + \hat{L}_i)^2 \\ &\quad - 2\mathbb{E} \sum_{i=1}^N (R_i - L_i)^2 (\hat{R}_i + \hat{L}_i). \end{aligned}$$

In the cases that $L_i(t)$ does not intersect with $R_j(t)$, it is easy to see that

$$\mathbb{E} \left[(L_i^2 - \hat{L}_i)(R_j^2 - \hat{R}_j) \right] = 0.$$

In the cases that $L_i(t)$ intersects with $R_j(t)$, let $Q(i, j) := L_i(t) \cap R_j(t)$, and let $L_{i,1}$ and $R_{j,1}$ respectively denote the parts of $L_i(t)$ and $R_j(t)$ that lie above $Q(i, j)$. Also, let $L_{i,2}$ and $R_{j,2}$ respectively denote the parts of $L_i(t)$ and $R_j(t)$ that lie below $Q(i, j)$. We can condition on $\mathcal{G}_{t,x_{j+1}} \vee \mathcal{G}_{t,x_{i+1}}$ to see that

$$\begin{aligned} &\mathbb{E} \left[(L_i^2 - \hat{L}_i)(R_j^2 - \hat{R}_j) \right] \\ &= \mathbb{E} \left[2 \int_{L_{i,2}} \sigma(u(v_i(y), y)) \xi(ds dy) \int_{Q(i,j)} \sigma(u(v_i(y), y)) \xi(ds dy) \right. \\ &\quad \left. + \left| \int_{L_{i,2}} \sigma(u(v_i(y), y)) \xi(ds dy) \right|^2 + \left| \int_{Q(i,j)} \sigma(u(v_i(y), y)) \xi(ds dy) \right|^2 \right. \\ &\quad \left. - \int_{L_{i,2}} [\sigma(u(v_i(y), y))]^2 ds dy - \int_{Q(i,j)} [\sigma(u(v_i(y), y))]^2 ds dy \right] \times \\ &\quad \times \left[\left| \int_{R_j(t)} \sigma(u(v_i(y), y)) \xi(ds dy) \right|^2 - \int_{R_j(t)} [\sigma(u(v_i(y), y))]^2 ds dy \right]. \end{aligned}$$

A few more rounds of conditioning on $\mathcal{G}_{t,x_j} \vee \mathcal{G}_{t,x_i}$ yield

$$\mathbb{E} \left[(L_i^2 - \hat{L}_i)(R_j^2 - \hat{R}_j) \right] := \mathbb{E} [(\mathcal{L}_{i,1} + \mathcal{L}_{i,2})(\mathcal{R}_{j,1} + \mathcal{R}_{j,2})],$$

where

$$\begin{aligned}
\mathcal{L}_{i,1} &:= 2 \int_{L_{i,2}} \sigma(u(v_i(y), y)) \xi(ds dy) \int_{Q(i,j)} \sigma(u(v_i(y), y)) \xi(ds dy) \\
&\quad + \left| \int_{Q(i,j)} \sigma(u(v_i(y), y)) \xi(ds dy) \right|^2 - \int_{Q(i,j)} [\sigma(u(v_i(y), y))]^2 ds dy, \\
\mathcal{L}_{i,2} &:= \left| \int_{L_{i,2}} \sigma(u(v_i(y), y)) \xi(ds dy) \right|^2 - \int_{L_{i,2}} [\sigma(u(v_i(y), y))]^2 ds dy, \\
\mathcal{R}_{j,1} &:= 2 \int_{R_{j,2}} \sigma(u(v_j(y), y)) \xi(ds dy) \int_{Q(i,j)} \sigma(u(v_j(y), y)) \xi(ds dy) \\
&\quad + \left| \int_{Q(i,j)} \sigma(u(v_j(y), y)) \xi(ds dy) \right|^2 - \int_{Q(i,j)} [\sigma(u(v_j(y), y))]^2 ds dy, \text{ and} \\
\mathcal{R}_{j,2} &:= \left| \int_{R_{j,2}} \sigma(u(v_j(y), y)) \xi(ds dy) \right|^2 - \int_{R_{j,2}} [\sigma(u(v_j(y), y))]^2 ds dy.
\end{aligned}$$

We may condition on $\mathcal{G}_{t,x_{i+1}} \cap \mathcal{G}_{t,x_j}$ in order to see that $E[\mathcal{L}_{i,2}\mathcal{R}_{j,2}] = 0$. Therefore, the Minkowski and Cauchy-Schwarz inequalities together show that

$$E(\mathcal{L}_{i,1}\mathcal{R}_{j,1} + \mathcal{L}_{i,1}\mathcal{R}_{j,2} + \mathcal{L}_{i,2}\mathcal{R}_{j,1}) \lesssim (x_{i+1} - x_i)^{3/2}(x_{j+1} - x_j) \lesssim N^{-5/2},$$

and hence

$$\sum_{1 \leq j < i \leq N} E(\mathcal{L}_{i,1}\mathcal{R}_{j,1} + \mathcal{L}_{i,1}\mathcal{R}_{j,2} + \mathcal{L}_{i,2}\mathcal{R}_{j,1}) \lesssim N^{-1/2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

In like manner, the Minkowski and Cauchy-Schwarz inequalities also show that

$$E \sum_{i=1}^N (R_i - L_i)^4 + E \sum_{i=1}^N (\hat{R}_i + \hat{L}_i)^2 - 2E \sum_{i=1}^N (R_i - L_i)^2 (\hat{R}_i + \hat{L}_i) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Thus, $\lim_{N \rightarrow \infty} E(|B_N(t) - C_N(t)|^2) = 0$, as was announced. Finally, it is possible to reuse the arguments of Steps 1–3 in order to show that $C_N(t) \rightarrow D(t)$ in L^p as $N \rightarrow \infty$. It was shown earlier in Step 4 that $C_N(t) - B_N(t) \rightarrow 0$ and $B_N(t) - A_N(t) \rightarrow 0$. Thus, $A_N(t) \rightarrow D(t)$ in $L^p(\Omega)$ as $N \rightarrow \infty$, as was desired. \square

3 Sketch of the Proof of Theorem 2

As in the proof of Theorem 1, define $r(y) := \max(t - |x - y|, 0)$, and set

$$\tilde{Q}(t, t + \varepsilon, x) := \{(s, y) : (s, y) \in Q(t, t + \varepsilon, x) \text{ and } |y - x| \leq t\},$$

where $Q(t, t + \varepsilon, x)$ was defined in (5). Next, define for all $h > 0$,

$$M_h = M_h(t, x) := \int_{\tilde{Q}(t, t+h, x)} \sigma(u(r(y), y)) \xi(ds dy),$$

and set $M_0 := \lim_{h \downarrow 0} M_h = 0$. The elementary properties of the Walsh stochastic integral imply that, given $\mathcal{G}_{t,x}$, the process $\{M_h\}_{h \geq 0}$ is conditionally a mean-zero, continuous $L^2(\Omega)$ -martingale with quadratic variation $\langle M \rangle_h = h\mathcal{V}$, where $\mathcal{V} = \mathcal{V}(t, x)$ is the $\mathcal{G}_{t,x}$ -measurable random variable,

$$\mathcal{V} := \int_{x-t}^{x+t} [\sigma(u(t - |x - y|, y))]^2 dy.$$

Thus, Lévy's characterization theorem of Brownian motion implies that, given $\mathcal{G}_{t,x}$, M is conditionally a Brownian motion with variance $h\mathcal{V}$ at time $h > 0$. As in the proof of Theorem 1 (see Steps 1 and 2 of that proof), one can prove that

$$\lim_{h \downarrow 0} \frac{u(t+h, x) - u(t, x) - M_h}{\sqrt{h}} = 0 \quad \text{in } L^2(\Omega).$$

We omit the details. Instead, we mention only that the central limit theorem (6) follows immediately from this and the scaling properties of the [conditional] Brownian motion M .

In order to prove the more delicate law of the iterated logarithm of the theorem, define

$$R_h = R_h(t, x) := u(t+h, x) - u(t, x) - M_h \quad \text{for all } h \geq 0.$$

It is not hard to use (8) together with the Burkholder–Davis–Gundy inequality and (9), as well as (10), in order to see that, for every $p \geq 2$, $\|R_h\|_p \lesssim h$ uniformly for all $h \in [0, 1]$. Since $R_0 = 0$, the Kolmogorov continuity theorem implies that, for every fixed $\eta \in (0, 1)$,

$$\left\| \sup_{s \in [0, h]} |R_s| \right\|_p \lesssim h^\eta \quad \text{uniformly for all } h \in [0, 1].$$

Thus, a standard application of the Borel–Cantelli lemma yields $R_h = o(h^\eta)$ almost surely as $h \downarrow 0$. In particular,

$$\limsup_{h \rightarrow 0} \frac{u(t+h, x) - u(t, x)}{\sqrt{2h \log \log(1/h)}} = \limsup_{h \rightarrow 0} \frac{M_h}{\sqrt{2h \log \log(1/h)}} = \mathcal{V}^{1/2} \quad \text{a.s.}$$

thanks to Khintchine's LIL for the [conditional] Brownian motion M . The law of the iterated logarithm of the theorem—see (7)—follows. \square

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